Determining the Magnetic Properties of 1 kg Mass Standards
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1. Introduction

Suppose that a mass standard having a small but finite magnetic susceptibility or permanent magnetization has been placed on one pan of a beam balance. The magnetic properties of the standard may lead to a potential energy term that also depends on the magnetic fields within the balance. If this magnetic potential energy changes as a function of rotation of the balance beam, then there will be an unwanted torque on the beam. One may also note that rotation of the beam through small angles is equivalent to motion of the mass standard in the vertical direction. From this qualitative argument, it is clear that a magnetic potential due to the mass standard and any balance components fixed with respect to the frame of reference of the standard can pose no problems to weighing. Problems, when they occur, are due to the magnetic properties of the mass standard and local magnetic fields that are in a frame of reference (usually that of the laboratory) that moves relative to the standard on the balance pan. In order to quantify such problems it is necessary to characterize both the magnetic properties of the mass standard and the magnetic environment of the laboratory. This paper will concentrate on the former task.

In a previous publication [1] we showed how the volume magnetic susceptibility $\chi$ of nonmagnetic materials could be measured by simple modification of a commercial microbalance. Tests carried out at the Bureau International des Poids et Mesures (BIPM) verified that a reasonably uncomplicated theory of operation is adequate to achieve results that have a relative combined standard uncertainty of a few percent over the range of susceptibilities encountered in mass standards of high quality. (Throughout this report uncertainties are expressed according to guidelines set forth by the International Organization for Standardization [2].)

In this report, we first discuss a model for errors in mass metrology due to magnetic effects. These are shown to depend mainly on the susceptibility and the permanent magnetization of the mass standard. The latter is zero for nonmagnetic materials but some materials used in the manufacture of good mass standards are, in fact, weakly magnetic and thus may become magnetized.

Susceptibility can be measured quantitatively using the BIPM apparatus and permanent magnetization can be detected. These measurements are discussed in
detail, including calibration of the instrument and estimation of measurement uncertainty. Of equal importance to making good measurements is interpreting their significance. We therefore begin with a basic review of the most probable magnetic problems in precision mass measurement and conclude with several practical examples.

2. Model for Magnetic Errors in Mass-Metrology

By magnetic error, we mean an unsuspected vertical force $F$ that is magnetic in origin. Such a force will be misinterpreted as a mass $F/g$, where $g$ is the local acceleration of gravity.

We may assume that high-quality mass standards are artifacts with an isotropic volume magnetic susceptibility $\chi$ whose magnitude is much less than one. In addition, artifact mass standards should have little or, ideally, no permanent magnetization. Thus the unwanted magnetic force will, to a good approximation, be given by

$$F = -\frac{\mu_0}{2} \frac{\partial}{\partial z} \int \chi' H \cdot H \, dV - \mu_0 M_z \frac{\partial}{\partial z} \int M \cdot H \, dV,$$  

where $\chi'$ is the effective volume magnetic susceptibility of the standard, $M$ is its permanent magnetization (defined as the magnetic moment per unit volume in zero field), $H$ is the local magnetic field strength and the $z$-axis is parallel to $g$. In general, all these terms may depend on position but we assume that $\chi'$ is a scalar. The parameter $\mu_0$ is the vacuum permeability, identically equal to $4\pi \times 10^{-7}$ N·A$^{-2}$. The integrals are taken over the volume of the artifact. The effective susceptibility $\chi'$ is defined as $\chi' = \chi_n$, where $\chi_n$ is the volume susceptibility of air ($4.3 \times 10^{-7}$). Because of its relatively small magnitude, $\chi_n$ can be neglected in most of the examples given below. The symbols, quantities and nomenclature used in this report are those set forth by the International Union of Pure and Applied Physics [3].

We have assumed in Eq. (1) that $H$ is the field before the sample is introduced, an approximation good to first order in the susceptibility. We have also assumed that the alloy is linear, i.e., its susceptibility is independent of applied magnetic field for strengths less than, say, 4 kA·m$^{-1}$. Finally, we have assumed that the effect of a permanent magnetization $M$ can simply be added as a term separate from the induced magnetization $\chi' H$. The validity of this approach will be demonstrated by examples given below. Note that, for a mass standard to be magnetized, it must have been exposed to field strengths greater than the linear threshold.

We can further simplify Eq. (1) by the additional assumption, usually warranted, that the linear and isotropic susceptibility is also homogeneous throughout the artifact. Finally, we make an unwarranted assumption that $M$ is a constant in magnitude and direction throughout the artifact and is independent of $H$ at low field strength. It would be difficult to proceed without the last assumption but the reader should keep in mind that its importance in what follows is largely heuristic. This allows us to write a simplified force equation:

$$F = -\frac{\mu_0}{2} \frac{\partial}{\partial z} \int H \cdot H \, dV = \mu_0 M_z \frac{\partial}{\partial z} \int H_z \, dV,$$

$$-\mu_0 M_t \frac{\partial}{\partial z} \int H_t \, dV.$$  

(2)

It should be noted that $M_t$, $M_r$ and $M$ are the vertical and horizontal components of $M$ and thus change with the orientation of the artifact. Without knowledge of the magnetic fields and gradients within the balance our model can take us no further except to imply the following: (i) There can be no magnetic errors if there are no field gradients in the balance, and (ii) Due to the symmetry of certain shapes (e.g., cylinders) some artifacts can be turned upside-down without changing the force contribution from the susceptibility. The sign of $M_t$ is, by contrast, reversed for these shapes.

Two serious attempts have been made to apply Eq. (2) to problems of mass metrology. Gould [4] presented several special cases to illustrate his contention that the worst problems are likely to occur when the mass standard and a part of the balance case (including the mass transporter in automated balances) located just below the pan are both accidentally magnetized along the vertical axis. This argument implies that the second integral in Eq. (2) is the most important contribution, the fields and gradients arising primarily from the magnetized parts of the balance.

Using this reasoning as a guide for the selection of suitable stainless steel alloys, Gould concluded that an alloy used for mass standards should be chosen both for its low susceptibility and for its resistance to permanent magnetization upon exposure to high fields. He found that for stainless steels, these two desirable properties are well correlated. That is, the alloy with the lowest susceptibility was also the most difficult to magnetize. The recommended alloy was found to have $\chi = 0.003$ when measured in a uniform field strength of 16 kA·m$^{-1}$. The permanent magnetization was less than 1 A·m$^{-1}$ after exposure to a "suitably large" uniform field (we have converted values given in Ref. [4] from CGS-EMU to SI [3]).

Kochsiek [5] approached the problem in a different way. He dealt with Eq. (2) by arguing that, once demagnetized, the normal use of mass standards should not
subject them to fields great enough to remagnetize them. A strong recommendation was made against selecting inferior stainless steel alloys, known to be easily magnetized. He then chose a special case in which the integrals in Eq. (2) may be solved trivially: the magnetic field within the balance chamber is vertical with the form

\[ H_z = h_0 + h_1 z, \]  

where \( h_0 \) and \( h_1 \) are constants and the origin of the coordinate system is at the geometric center of the mass standard. As an example, \( h_0 \) was chosen to be 100 A·m⁻¹ and \( h_1 \) to be 5000 A·m⁻². (As a point of reference, the vertical component of the Earth’s magnetic field strength is about 40 A·m⁻¹ at the latitude of Paris.) Based on this somewhat arbitrary choice of “worst-case” parameters, it was then possible to argue that secondary 1 kg mass standards used by national laboratories should have a volume susceptibility below 0.003.

It is instructive to note that this recommendation also allows us to calculate an upper limit for \( M_z \) in Eq. (2):

\[ 1/M_z < 0.3\text{ A·m}^{-1}. \]  

The limit is derived from Eqs. (2) and (3) with the condition that the second integral in Eq. (2) be smaller than the first when \( \chi' < 0.003 \). In general for this model, \( M_z \) is negligible when \( 1/M_z < 1.1\chi' h_0 \).

Thus we see that Refs. [4] and [5], while focusing on different aspects of the problem, arrive at nearly the same guidelines for selecting stainless steel alloys suitable for the highest quality of secondary mass standards. Nevertheless, it should be emphasized that Eq. (2) cannot be solved without detailed knowledge of the fields within the balance. The Organisation Internationale de Métrologie Légale (OIML) has recently recommended that Class E₁ and E₂ mass standards have volume susceptibilities less than 0.01 and 0.03 respectively [6].

3. Susceptometer Developed at the BIPM

The BIPM susceptometer subjects a mass standard to relatively small, calculable fields, measures the resulting vertical force and then makes use of Eq. (2) to find the susceptibility and the parameter \( M_z \). The principles of construction are given in Ref. [1] but we show the apparatus schematically in Fig. 1 and give design details in Appendix A. The magnet that we use is cylindrical with height \( L \) and diameter both equal to 5 mm.

The balance reads in units of mass, the reading becoming more negative when a paramagnetic sample is introduced. Thus, the following relation is used:

\[ +1 \mu\text{N corresponds to a balance reading of } -101.9 \mu\text{g}, \]

where the acceleration of gravity \( g \) has been taken to be 9.81 m·s⁻².
1. However, the usual shape of cylindrical rare-earth magnets has $\gamma = 0.5$, so it is convenient simply to combine two such magnets.

Using the property of superposition, we can immediately find the force on a slab of finite thickness $t$. If we define $Z_i = Z_0 + t$, then

$$F = \chi' F_{\text{max}} \left[ 1 - \left( \frac{Z_0}{Z_i} \right)^4 \right] = \chi' F_{\text{max}} \left[ 1 - Z_i^4 \right]$$

where the dimensionless quantity $Z_\text{in}$ is the length $Z_i$ normalized to $Z_0$.

More generally, we next consider how large a cylindrical sample of radius $a$ and height $t$ must be before it may be approximated as "semi-infinite." To do this, we insert the dipole fields due to the magnet in the first integral of Eq. (2) and solve for the case of cylindrical samples of finite dimensions. The result of the calculation is given in Eqs. (5) and (6).

Figure 3 shows contours of equal force between the magnet and samples of differing dimensions. The force is normalized to $\chi' F_{\text{max}}$ and the cylinder dimensions are normalized to $Z_0$. Only a finite volume of semi-infinite samples contributes strongly to the force integral. The important volume is of order $Z_0^3$ and thus changes as $Z_0$ is varied experimentally.

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**Fig. 2.** Expected behavior of three cylindrical magnets of different aspect ratio ($\text{height/diameter}$) $\gamma$ relative to a spherical magnet. All four magnets have the same volume ($98 \text{ mm}^3$) and uniform magnetization. The case considered is the attractive force between each magnet and a semi-infinite slab of small susceptibility. The spherical magnet is taken as the reference and is thus represented by the abscissa.

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**Fig. 3.** Contours of constant force between a dipole magnet and coaxial cylinders of varying radius $c_n$ and thickness $t_n$ but having the same, small susceptibility. Cylinder dimensions are normalized relative to $Z_0$, the distance between the center of the magnet and the base of the cylinders. Force contours are normalized to the signal from a cylinder of infinite radius and height (i.e., a semi-infinite slab).
3.2 General Equations

A sample placed on the susceptor as shown in Fig. 1 will be subjected to fields from the cylindrical permanent magnet \((H_{\text{max}})\) and the Earth \((H_{\text{E}})\). Making the dipole approximation for the cylindrical magnet and taking the Earth’s field to be uniform leads to the following force equation \([1]\)

\[
F = \chi' F_{\text{max}} I_s + \frac{\mu_0}{4\pi} (\chi'H_{\text{E}} + M_z) \frac{m}{Z_0} I_b = F_x + F_y, \quad (5a)
\]

where

\[
\begin{align*}
I_s &= -\frac{32\pi}{3m^2} \frac{\partial}{\partial Z_0} \int \int_{V_s} (H_{\text{max}} \cdot H_{\text{max}}) \, dV = \\
&= -\frac{2}{3\pi} \frac{\partial}{\partial Z_0} \int \int_{V_s} \left( \frac{\rho^2 + 4z^2}{\rho^2 + z^2} \right) \rho \, d\rho \, d\theta \, dz , \quad (5b)
\end{align*}
\]

\[
I_b = -\frac{4\pi}{m} \frac{\partial}{\partial Z_0} \int \int_{V_s} (H_{\text{max}}), \, dV = \\
&= -\frac{\partial}{\partial Z_0} \int \int_{V_s} \left( \frac{\rho^2 - 2z^2}{\rho^2 + z^2} \right) \rho \, d\rho \, d\theta \, dz \quad (5c)
\]

and \(F_{\text{max}}\) is defined in Eq. (4). We have assumed the samples to be centered symmetrically about the magnet axis so that only the vertical components of \(H_{\text{E}}\) and \(M\) contribute to the force. The integrals are taken over the normalized volume of the sample. It is evident from Eq. (5a) that the induced magnetization \(\chi'H_{\text{E}}\) has the same effect as a permanent magnetization \(M_z\). We show below how the two contributions may, in practice, be distinguished. The differentiated integrals of Eqs. (5b) and (5c) are shown in detail for cylindrical coordinates where the origin is, as always, at the center of the magnet.

In practice, an initial force measurement \(F_x\) is made with the north pole of the magnet pointing down and a second measurement \(F_y\) is made (at the same \(Z_0\)) with the north pole pointing up. Then the first term in Eq. (5a) is given by \(F_x = (F_x + F_y)/2\) and the second term by \(F_y = (F_x - F_y)/2\). It was noted in Ref. [1] that \(Z_0\) changes slightly each time the magnet is repositioned. In practice, we have not found this to be a problem for nominal settings of about 15 mm or greater.

From a measurement of \(F_x\) using the unknown sample and a knowledge of \(m, Z_0\) and the sample dimensions, we can solve for \(\chi'\), using Eqs. (4), (5a) and (5b). Using this \(\chi'\), the measured value of \(F_y\), and a knowledge of \(H_{\text{E}}\), we can then find \(M_z\) from Eqs. (5a) and (5c). The uniform magnetization is usually negligible. The quantities \(I_s\) and \(I_b\) are readily written in closed form for cylindrical samples of radius \(c\) and height \(t\) coaxial with the magnet. This is because each differentiated triple integral reduces to a single integral in this case. We repeat the relations, given in Ref. [1], using normalized variables\(^2\):

\[
\begin{align*}
I_x &= 1 - Z_{\text{in}}^4 - \frac{1 + c_n^3/3}{(1 + c_n^3)^3/2} + Z_{\text{in}}^4 \frac{1 + (c_n/Z_0)^3/3}{[1 + (c_n/Z_0)^3]^{3/2}} \quad (6a)
\end{align*}
\]

and

\[
I_y = 2\pi \left[ \frac{c_n^3}{(1 + c_n^3)^3/2} - \frac{c_n^2 Z_0}{[1 + (c_n/Z_0)^3]^{3/2}} \right] . \quad (6b)
\]

where \(Z_{\text{in}} = 1 + t_n\). Note that \(0 \leq I_x, I_y \leq 1\), the upper limit being the case of a semi-infinite slab and the lower limit a sample of vanishingly small volume. In contrast, \(0 \leq I_y \leq 2.42\), the lower limit being obtained for both a semi-infinite slab and a sample of vanishingly small volume. For shapes other than cylinders, solutions can be found by numerical integration and/or superposition of easily calculable shapes. Several examples are given below and in Appendices B and C.

It can be shown in general that

\[
\frac{F_x}{F_y} = \frac{3\pi \chi'H_{\text{max}} I_s}{8 \chi'H_{\text{E}} + M_z I_b} ,
\]

where

\[
H_{\text{max}} = \frac{m}{2\pi Z_0} ,
\]

the magnitude of the maximum field strength to which the sample is subjected by the magnet alone. This is the field at the base of the sample, directly above the magnet. An interesting special case, which is not unusual, is

\(^1\) This assumes that the laboratory is not magnetically shielded and that no extraneous fields are present. These assumptions were verified for our laboratory.

\(^2\) Equation (8) of [1] has obvious typesetting errors.
that of a cylinder for which \( Z_{\text{in}}^4 \gg 1 \) and \((Z_{\text{in}}/c_0)^2 \gg 1\). Then \( I_0\) takes its maximum value at \( c_0 = \sqrt{2} \), in which case \( I_0/\mu_0 = 0.4 \). Thus we see that, for a given \( Z_0 \), an apparatus realized with a much more sensitive balance and a much smaller magnet than we use would favor the measurement of \( F_0 \) instead of \( F_\alpha \).

At any given setting of \( Z_0 \), the geometric terms in Eq. (5) are the same for samples having the same dimensions and orientation. Thus the relative susceptibilities of congruent samples are easily determined from ratios of balance readings. To determine an unknown susceptibility, one might even machine a material of known susceptibility to match the dimensions of the unknown object [7]. Unless a large number of identical samples is to be determined, or the field of the magnet cannot be approximated as a dipole, this approach is less convenient than finding sufficiently accurate solutions to the differentiated integrals of Eqs. (5b) and (5c).

The susceptibilities of a set of objects, each having the same shape and size (such as similar mass standards), may be conveniently determined by determining the susceptibility of one and then treating it as a standard to find the susceptibility of the others through ratios of the observed values of \( F_\alpha \).

It now remains to show how the BIPM susceptometer may be calibrated and how we use it.

### 4. Calibration of Susceptometer

Calibration of the device is equivalent to determining \( m \) and \( Z_0 \). Several methods may be used and they give equivalent results. The two unknown parameters may, for example, be found by measuring two standards of different, known susceptibility. Such standards are not readily available in a suitable range of susceptibilities, however [8]. A reasonably good estimate of \( m \) can be inferred from technical information supplied with the magnet. As shown below, the susceptometer may also be self-calibrated by fitting measured values of \( F_\alpha \) as a function of increments in \( Z_0 \), obtained using a sample of unknown susceptibility, to the equations developed above. All these methods should, of course, be consistent within their combined uncertainties.

#### 4.1 Determination of \( m \) From Supplier's Specifications

At distances \( Z_0 \) that are large compared with the magnet dimensions, the axial magnetic field strength due to the magnet is given by \( H_{\text{max}} \), and may be positive or negative depending on the orientation of the magnet. If the magnet has a uniform axial magnetization \( M \), then \( m = MV \) where \( V \) is the volume of the magnet. It is more usual to characterize a permanent magnet by its polarization \( J = \mu_0 M \), which has units of tesla.

The supplier of our magnets provided a depolarization curve that shows how \( J \) is reduced in the presence of an opposing external field (Fig. 4). The shape of the magnet also has an effect on \( J \). Even with no applied field, a cylindrical magnet will suffer to some extent from self-demagnetization if \( \gamma < \infty \). In the case of a cylindrical magnet with \( \gamma = 1 \) (height equal to diameter), the demagnetizing factor \( N_m \) is 0.312 [9]. Thus a first estimate of \( m \) is the volume of the magnet multiplied by the value of \( M \) corresponding to the dot in Fig. 4. Note that the demagnetizing factor of the magnet has a relatively small effect on \( M \). This is an important property of rare-earth magnets.

![Depolarization curve](image)

**Fig. 4.** Depolarization curve supplied by the manufacturer of the magnet used in the susceptometer. The ordinate represents the magnetic polarization and the abscissa the magnetic field strength opposing the polarization. For a cylindrical magnet with height equal to diameter \( (N_m = 0.312) \), self-demagnetization shifts the operating point from the \( J \)-intercept (no opposing field strength) to the point indicated by a dot.

Taking \( M \) and \( V \) from the manufacturer’s data sheets thus gives us the following estimate: \( m = 0.0898 \, \text{A} \cdot \text{m}^3 \). It is difficult to assign an uncertainty but we will see in Sec. 4.2.2 that experimental values are within 1.5% of this.

- The estimated value of \( m \) is useful in computing an approximate value for \( H_{\text{max}} \), the maximum field strength to which the sample will be exposed at a nominal setting of \( Z_0 \). For 15 mm < \( Z_0 \) < 30 mm, 4.25 \( \text{kA} \cdot \text{m}^{-1} \) > \( H_{\text{max}} \) > 0.53 \( \text{kA} \cdot \text{m}^{-1} \).
4.2 Bootstrap Calibration

The calculation of \( m \) given in Sec. 4.1 is a good approximation but the definitive value for \( m \) is best determined experimentally. In any case, we still require a routine method of finding \( Z_0 \) to sufficient accuracy. Both needs can be met by carrying out an initial series of bootstrap measurements, shown schematically in Fig. 5 and described in detail below. We first determine \( \chi_0', m^2 \) for a selected standard \( S \). Based on Eqs. (4) and (5a), this quantity then allows us to find \( Z_0 \) by placing the standard on the apparatus and measuring \( F_0 \). Once we have a way of determining \( Z_0 \), we can find \( m' \), as shown below. The bootstrap measurements require considerable effort but need only be done once. In the following, we analyze statistical uncertainties (i.e., Type A standard uncertainties) only. Type B standard uncertainties are discussed in Sec. 4.3.

4.2.1 Determination of \( \chi_0', m^2 \) (and hence \( Z_0 \)) Using gauge blocks (see Appendix A), \( Z_0 \) can be incremented in precise steps of, for example, 5 mm. The value of \( Z_0 \) may thus be known to within a constant length \( Z_{\infty} = Z_0 - Z_0 \), where \( Z_0 \) is the height of the gauge blocks in use. If we had a suitable standard with known susceptibility \( \chi_0 \) and if, further, we knew \( m^2 \), then a measurement of \( F_0 \) using the standard would determine the value of \( Z_0 \) each time the height of the stage was changed.

Standard samples of sufficiently large susceptibility are notoriously difficult to obtain [8] and so we standardize an unknown sample by measuring it at different settings of \( Z_0 \) and fitting the results to the function predicted by the theory presented above. The parameters obtained from this fit contain some of the information we require. The force \( F_0 \) computed from Eqs. (4), (5a) and (6a) depends on the dimensions of the standard as well as the quantities \( (\chi_0', m^2) \) and \( Z_0 \). For a suitable standard which is linear (up to field strengths of about 20 kA\cdotm^{-1}), isotropic and homogeneous, \( F_0 \) as a function of \( Z_0 \) may be fitted using two adjustable parameters and hence the unknowns \( \chi_0', m^2 \) and \( Z_0 \) can be found. In the following we will use the abbreviation LIH to refer to materials that are magnetically linear, isotropic and homogeneous. As long as we use the same magnet and the same sample, \( \chi_0', m^2 \) will not, in the normal course of events, change with time and so the standard may be used to find \( Z_0 \) whenever the height of the apparatus is changed.

Given a cylindrical sample that is LIH and has a susceptibility of about 0.0015, the major difficulties are: i) The computation of \( I_0 \) requires a value for \( Z_{\infty} \), which is unknown. Thus all calculations must be carried out iteratively by making an initial guess of \( Z_{\infty} \).

Fig. 5. Flow charts showing the two steps of self-calibration. Step 1 determines the quantity \( \chi_0', m^2 \) and Step 2 then determines the individual parameters \( m \) and \( \chi_0' \). Computations shown in Step 1 can be greatly simplified through the use of commercial software.
ii) The functional form of \( F_s \) versus \( Z_b \) is not linear. A linear result will be obtained by plotting \((I_s/F_s)^{1/4}\) versus \( Z_b \).

iii) As shown in Sec. 3.1, results obtained at relatively small values of \( Z_b \) are biased by assuming the cylindrical magnet to be a dipole.

The experimental algorithm is shown as Step 1 of Fig. 5. The sample we chose is made of the nonmagnetic alloy Alacrite X.S.H. (Aubert and Duval S. A., Neufly sur Seine, France) whose nominal mass composition is: 20 \% Cr, 15 \% W, 10 \% Ni, 0.1 \% C, remainder Co. It is a polished disc with a diameter of 69 mm and a thickness of 9.8 mm and had already been fabricated for another purpose. In retrospect, a thickness of about 20 mm would have been preferable to increase the signal. Measurements of \( F_s \) were made starting at \( Z_b = Z_{b0} \approx 9.5 \) mm and in precise increments of 5 mm up to \( Z_b = Z_{b0} + 20 \) mm.

Table 1 shows the final iterated values of \( \chi', m^2 \) and \( Z_{b0} \) based on three different least squares fitting routines. For fit (1), a weighted linear least-squares routine was used, where the uncertainty in \( Z_b \) is assumed to be negligible. Formally, we are fitting \( y = -\beta_1 x + \beta_0 \) where \( y = (I_s/F_s)^{1/4} \) and \( x = Z_b \). Since \( I_s \) depends weakly on \((\beta_1/\beta_0) + x\), several iterations are required.

Table 1. Data and data analyses for determination of the quantity \( \chi', m^2 \) from measurements of \( F_s \) as a function of \( Z_b \). Standard uncertainties for the measurements of \( F_s \) are given in the second column.

<table>
<thead>
<tr>
<th>( F_s / \mu N )</th>
<th>( u_0/\mu N )</th>
<th>( Z_{b0} / )mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.65</td>
<td>0.07</td>
<td>0</td>
</tr>
<tr>
<td>3.990</td>
<td>0.0012</td>
<td>5</td>
</tr>
<tr>
<td>1.112</td>
<td>0.010</td>
<td>10</td>
</tr>
<tr>
<td>0.4042</td>
<td>0.005</td>
<td>15</td>
</tr>
<tr>
<td>0.1702</td>
<td>0.005</td>
<td>20</td>
</tr>
</tbody>
</table>

- \( \beta_1 \mu N \) corresponds to a balance reading of \(-101.9 \) \( \mu g \).
- \( \beta_0 \mu N \) is determined from the table result from different ways of manipulating the same data. Fit (3), which was carried out using commercial software, is taken as the definitive result.

\( ^{+} \) Certain commercial equipment, instruments, or materials are identified in this paper in Partial Progress. Such identification does not imply recommendation or endorsement by the National Institute of Standards and Technology, nor does it imply that the materials or equipment identified are necessarily the best available for the purpose.

We note that using the fitted values to extrapolate to \( Z_b = 0 \) results in a value of \( F_s \) that is about 5 \% lower than the observed force. The implications of this error are discussed in Sec. 4.3.

4.2.2 Determination of \( m \) Given \( \chi', m \), it is a relatively simple step to determine \( m \), and hence \( \chi' \), (Step 2 of Fig. 5). To do this, we need two additional magnets, B and C, of similar dimensions to A, the magnet we normally use. The additional magnets have moments that are, as yet, undetermined although all three moments are expected to be consistent with the estimate obtained in Sec. 4.1. We determine the magnetic moment of each magnet by using the susceptorometer to measure the force between all possible pairs of magnets, aligned coaxially and placed a known distance apart.

We first set the span of the susceptorometer so that \( Z_0 \) is approximately 15 mm. A precise value of \( Z_0 \) is determined from a measurement of \( F_s \) using the Alacrite standard whose calibration was described in Sec. 4.2.1. We then add gauge blocks with a height \( Z_b = 70 \) mm so that \( Z_0 \) is nominally 85 mm.

At this setting, we measure the force between magnets A and B. Magnet B is placed on the span so that the resulting interaction force is attractive. Its orientation is then reversed so that the reaction force is repulsive. The same sequence of measurements is repeated for C. Ensuring that the two magnets are sufficiently coaxial is relatively straightforward because the coaxial condition gives a maximum change in the balance reading.
The average magnitudes of the attractive and repulsive forces give us $F_\rho$ between magnets A, B and A, C. We then place magnet B on the pedestal and repeat the measurements by placing A and C on the span in each orientation to obtain $F_\rho$ between B, C and B, A. The last measurement serves as a check on the reproducibility of the results. As may be inferred from Table 2, we found the balance readings were nominally 8300 µg. The mean standard deviation of four repeated balance readings was 15 µg.

The force measured between magnets A and B is

$$ (F_\rho)_{AB} = F_{AB} = \frac{6 \mu_0}{4 \pi} \frac{m_A m_B}{(Z_0 + L/2)^4}, $$

where $L/2$ is 2.5 mm, half the height of the magnet, and we have added subscripts to distinguish the moments of magnets A and B. The equation is the well known relation for the interaction force between two coaxial dipoles and is a special case of Eqs. (5a) and (6b) when $Z_0$ is large compared with the sample dimensions and $\chi$ is negligible. Thus

$$ m^2 = F_{AB} \cdot F_{BC} \cdot \frac{4 \pi (Z_0 + L/2)^4}{6 \mu_0}. \quad (7) $$

The indices A, B, and C in Eq. (7) can be permuted in obvious fashion to obtain the moments of magnets B and C.

The results are given in Table 2. We note that, on average, the three magnetic moments agree well with the estimate of 0.0898 A·m² obtained in Sec. 4.1.

### Table 2. Calculation of magnetic moments $m$ of magnets A, B and C based on the force $F_\rho$ measured between pairs of magnets aligned at a spacing of 84.90 mm

<table>
<thead>
<tr>
<th>Magnets</th>
<th>$F_\rho/\mu N^*(\text{measured})$</th>
<th>$m/(A\cdot m^2)$ (calculated)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B</td>
<td>81.80</td>
<td></td>
</tr>
<tr>
<td>A, C</td>
<td>83.35</td>
<td></td>
</tr>
<tr>
<td>B, C</td>
<td>83.50</td>
<td></td>
</tr>
<tr>
<td>A, B</td>
<td>81.86</td>
<td>0.08910</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td>0.08937</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>0.09089</td>
</tr>
</tbody>
</table>

*+1 µN corresponds to a balance reading of -101.9 µg.

With the value of $m$ in hand, we can now compute $\chi'$, for the Alacrite standard. The result is

$$ \chi' = 0.001348 $$

with $\chi = 0.00012$ (Type B standard uncertainties neglected).

### 4.3 Calibration Checks

Once we have a standard with a known susceptibility, any height $Z_0$ is determined by a measurement of $F_\rho(s)$, where (s) denotes use of the standard. The susceptibility $\chi'$ of an object of interest is then determined through a measurement of the force $F_\rho$ due to the unknown object placed at the same $Z_0$. In essence, the following relation is used:

$$ \frac{\chi'}{\chi} = \frac{F_\rho}{F_\rho(s)} \frac{I(s)}{I_a} \quad (8) $$

where $I(s)$ is appropriate to the standard and $I_a$ to the unknown [Eqs. (5b) and (6a)]. In Sec. 3.2 we referred to the special case where the standard and unknown have the same dimensions so that $\{I(s)/I_a\} = 1$ in general. This relation also holds for "large" samples as defined in Sec. 3.1. The ratio of susceptibilities is manifestly insensitive to details of the theory for such cases. It is only when the term $\{I(s)/I_a\}$ depends sensitively on $Z_0$ that we must rely on the quantitative validity of the theory developed above.

Two measurements described in detail in Ref. [1] provide checks on the method of calibration used in the previous section and help in the assessment of overall Type B standard uncertainties. First, it was shown that the susceptibility measured for a number of well-characterized materials is generally in good agreement with handbook values. By itself, this is not a sufficient check of the apparatus because the susceptibilities of the standard materials used are relatively small in magnitude compared with the susceptibilities of stainless steels and other alloys of interest. Thus the standard materials cannot only be used to verify operation at relatively high fields, corresponding to values of $Z_0$ in the range of 10 mm to 15 mm. In this range, the approximation of the magnet as a dipole is expected to bias results for samples that are not "large." We expect good results for large samples, however, and this serves to check our inferred value of $\chi'$.

The second test is that the ratio of the measured susceptibilities of two different LIH alloys having significantly different dimensions and susceptibility is found to be independent of $Z_0$ through a range of settings where $\{I(s)/I_a\}$ changes by 50%. These tests suggest that, for the examples given below, the Type B relative standard uncertainty is no greater than 3% in the measured susceptibility for LIH samples.
5. Typical Use of Susceptometer

We find it convenient to use the automatic-zero feature of the balance. The reading displayed is consequently insensitive to small, slowly changing forces. We therefore take the following precautions during the measurements. We do not collect data at settings where the introduction of the sample changes the balance reading by less than 10 counts in the last displayed digit. To change the position of the sample, we remove it entirely, let the balance return to zero and then place the sample in its new position. We repeat a measurement if the balance zero changes significantly during a sequence of readings.

For routine measurements, we begin with $Z_o$ at no less than 25 mm and only decrease $Z_o$ if the sample susceptibility is too small to produce a reasonable signal. Setting $Z_o$ to 25 mm corresponds to a maximum field strength of $H_{max} = 900$ A m$^{-1}$ ($\mu m H_{max} = 1$ mT).

Appendices B and C summarize susceptibility measurements for typical 1 kg mass standards of Class E2. The first (Appendix B) has the shape of a right-circular cylinder with rounded edges and the second (Appendix C) has the external dimensions of a Class M standard [6].

5.1 Cylindrical Mass Standard

In the example shown in Appendix B, the sample was placed coaxial with the magnet at a nominal spacing of 30 mm. Measurements were made using both vertical orientations of the magnet and both vertical orientations of the sample. Data obtained using the Alacrite standard were used to determine the average $Z_o$, 30.55 mm. Equations (5a) and (6b) predict that the Alacrite standard should produce a value of $F_{o} / g = +5 \mu g$. We observe $46 \mu g$, which is a useful check that the device is operating properly.

As shown in Appendix B, the balance readings obtained when the bottom of the sample was nearer the magnet differ from those when the usual orientation of the sample is reversed. This could be a real difference due to sample inhomogeneity. However, the fact that $F_{o}$ is essentially identical for each orientation of the sample suggests that permanent magnetization is the more probable explanation.

We emphasize that our calculation of $M_{c}$ assumes a model of uniform magnetization. Evidence that this model is unrealistic is that the fitted value of $M_{c}$ for the sample in its normal orientation is 0.2 A m$^{-1}$ but 0.07 A m$^{-1}$ when the sample is reversed.

Based on the limit derived in Sec. 2, we would not expect the observed magnetization to be of concern. To see if the sample magnetization has any practical consequences, we measured the mass of the secondary standard using a magnetically servocontrolled balance (Mettler-Toledo HK 1000 MC). The results obtained were the same when the standard was upside down on the balance. This means that, although the sample has a detectable permanent magnetization, routine weightings in our laboratory should be unaffected.

5.2 OIML-Shaped Mass Standard

Mass standards are not generally cylinders, although they do possess cylindrical symmetry. A typical shape is the so-called OIML design [6], shown in Appendix C. It differs from a simple cylinder primarily in that it has a lifting knob on top and a recessed base.

Appendix C presents calculations of $\chi^2$ for a 1 kg standard in its usual orientation carried out at a setting of $Z_o$ typical for routine measurements. Both orientations of the magnet are used in order to obtain the value of $F_{o}$. The same data are used to infer an apparent value of $M_{c}$ from the calculation of $F_{o}$, see Eq. (5).

To demonstrate the generality of the theory, we also made measurements with a similar mass standard placed on its side so that its body was centered directly over the magnet. Equation (5b) was solved numerically, with limits of integration corresponding to the cylindrical body of the standard. As shown in Table 6 (Appendix C), the presence of the knob has a relatively small effect on the calculation. Experimental results were unchanged when the sample was rotated about its axis of symmetry.

6. Behavior of Nonlinear and/or Inhomogeneous Samples

The previous section described typical measurements for alloys used in the manufacture of good secondary mass standards. We now give examples of results obtained from test objects that are nonlinear and/or inhomogeneous.

6.1 Nonlinear Alloy, Initially Unmagnetized

Pure brass is diamagnetic. We find that much of the industrial brass supplied to our workshops is, on the contrary, contaminated with magnetic impurities to the extent that the measured susceptibility is frequently greater than 0.01 in field strengths ranging up to about 1 kA m$^{-1}$. Typically, the susceptibility depends on maximum field strength.

If one attempts to make the measurement at a maximum field strength of 10 kA m$^{-1}$ or more, the balance reading does not stabilize in the usual settling time but
continues to increase in magnitude for many minutes. This is an indication that a portion of the sample is becoming permanently magnetized, as can be verified by carrying out further measurements with a maximum field an order of magnitude less strong. Exposure of poor-quality samples to high fields must, therefore, be avoided. This is the reason we begin all routine testing at low fields. For the same reason, we take great care that test objects do not accidentally come in contact with the permanent magnet.

### 6.2 Nonlinear Alloy, Initially Magnetized

It is not unusual to encounter samples of industrial-grade copper alloys or of type 304 stainless steel that have accidentally been magnetized at some point in their history. Once magnetized, stainless steels are difficult to demagnetize, as is well known [5, 10].

We found one disc of copper-beryllium alloy (mass fraction of beryllium nominally 2 %) that was magnetized to such an extent that the term in \( M_\alpha \) in Eq. (5a) dominated the measurements. Results for this sample are given in Table 3. It may be seen that the model of uniform magnetization fits reasonably well in this case, with \( M_\alpha \) about 25 A·m\(^{-1}\). Commercial alloys of Cu-2 %Be contain small but significant amounts of ferromagnetic elements. It is, therefore, the heat treatment given to the alloy that renders it more or less magnetic [11].

We then passed the sample through a demagnetizing coil energized at the main frequency (50 Hz). Results obtained after this operation are also given in Table 3. We see that, although still present, the permanent magnetization is weaker and evidently not as uniform. Data obtained at the two nearest settings of \( Z_0 \) now reveal a measurable \( F_\alpha \), indicating a susceptibility of order +0.000.35. This value is consistent with the range of susceptibilities common to commercial alloys of Cu-2 %Be [11].

#### 6.3 Good-Quality Alloy, Surface Inhomogeneity

One of the well-characterized materials that we tested was a disc of oxygen-free copper with a mole fraction for iron impurity of only \( 2 \times 10^{-6} \). This alloy has a handbook susceptibility of \(-9.6 \times 10^{-8}\) [11]. The disc-shaped sample supplied by the manufacturer had rough-sawn faces that we machined flat before the measurements.

The first susceptometer reading appeared to show that the sample was paramagnetic. After it was given a light etch in a copper-cleaning solution, further measurements showed the sample to have the expected susceptibility. Thus the initial problem was a surface effect, presumably due to iron impurities transferred from the cutting tool. In order to have sufficient signal, these measurements were made at \( Z_0 \approx 10 \) mm.

Hard-working of stainless steel surfaces, as may happen during the fabrication of mass standards, can cause the susceptibility to increase [4]. Should this occur, the fabricated standard becomes inhomogeneous and use of Eq. (5b) will make it appear that the susceptibility of the sample is decreasing with increasing \( Z_0 \). This may be the explanation of results reported in Ref. [11] for a cylinder of the stainless steel Immaculate V where the susceptibility inferred from Eq. (5) decreased by about 7 % as \( Z_0 \) was increased from 9.3 mm to 24 mm. The susceptibility did not decrease significantly when \( Z_0 \) was further increased to 29 mm. The sample had no detectable permanent magnetization.

---

**Table 3.** Effect of “degaussing” on a magnetized disc of commercial-grade Cu-2 %Be. The influence of the Earth’s field on the calculation of \( M_\alpha \) may be neglected.

<table>
<thead>
<tr>
<th>Nominal ( Z_0/\text{mm} )</th>
<th>Measured ( F_\alpha/\mu\text{N} )</th>
<th>( M_\alpha/(\text{A}\cdot\text{m}^{-1}) )</th>
<th>Measured ( F_\alpha/\mu\text{N} )</th>
<th>( \chi/10^{-4} )</th>
<th>Measured ( F_\alpha/\mu\text{N} )</th>
<th>( M_\alpha/(\text{A}\cdot\text{m}^{-1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>12.68</td>
<td>25.5</td>
<td>0.96</td>
<td>3.52</td>
<td>7.51</td>
<td>13.2</td>
</tr>
<tr>
<td>20</td>
<td>10.06</td>
<td>25.2</td>
<td>0.27</td>
<td>3.40</td>
<td>5.80</td>
<td>12.1</td>
</tr>
<tr>
<td>25</td>
<td>7.72</td>
<td>24.9</td>
<td>4.395</td>
<td>24.4</td>
<td>3.14</td>
<td>10.7</td>
</tr>
<tr>
<td>30</td>
<td>2.531</td>
<td>24.0</td>
<td>0.963</td>
<td>24.0</td>
<td>0.97</td>
<td>9.7</td>
</tr>
<tr>
<td>70</td>
<td>0.404</td>
<td>22.8</td>
<td></td>
<td></td>
<td>0.37</td>
<td>9.7</td>
</tr>
</tbody>
</table>

\( +1 \mu\text{N} \) corresponds to a balance reading of \(-101.9 \) μg.
We cannot take quantitative account of an enhanced surface susceptibility without prior knowledge of either the thickness of the enhanced layer or its susceptibility.

7. Magnetic Fields in a Balance

It is possible to measure both the vertical magnetic field intensity and its gradient in the weighing chamber of the MT5 balance used for these measurements. Using a conventional magnetometer, we find the vertical field within the weighing chamber (with the cylindrical magnet removed) is about 10% greater than that of the Earth alone.

The gradient of this field, measured by raising the probe a few centimeters, was inferred to be of order 40 A·m⁻². An independent measure of the gradient is obtained by observing the change in balance reading when the magnet is reversed with no sample present. The measured difference is 1.5 mg implying a gradient of 65 A·m⁻² (note that the difference in balance reading upon reversal of the magnet reported in Ref. [1] was 0.9 mg; this was obtained from a different unit of the same model balance).

Thus, based on Eqs. (2) and (3), if the balance is used to weigh 5 g of stainless steel, the susceptibility of the material should be less than 5 and any uniform magnetization in the vertical direction should be less than 200 A·m⁻¹ in order to keep magnetic effects below the balance resolution of 1 μg. If the same average field strength and gradient were present in a 1 kg mass comparator, stainless steel standards would require $\chi < 0.025$ and $|M| < 1$ A·m⁻¹ in order to keep magnetic effects below 1 μg.

8. Conclusion

We have discussed how magnetic forces may lead to errors in mass measurements. The force equations that describe the unwanted effects also describe the operation of the BIPM susceptometer. We have presented design details of this device, which is suitable for checking the susceptibility of 1 kg mass standards of stainless steel and similar nonmagnetic or weakly magnetic alloys.

In addition, we have demonstrated that the device may be calibrated by using a set of gauge blocks, two additional magnets, and a LiH sample of unknown susceptibility. The calibration was verified by measurements, in relatively high field strengths, of materials having well-characterized susceptibilities.

The device can also detect a permanent magnetization. An apparent magnetization can be calculated and has a simple interpretation for uniformly magnetized samples. An acceptable upper limit for uniform magnetization has been derived in the spirit of a previous discussion of an acceptable upper limit for the susceptibility.

9. Appendix A. Construction of the Apparatus

The apparatus is shown schematically in Fig. 1 and the most important construction details are given below.

9.1 Balance

The balance used (Mettler-Toledo AG, model MT5) has a 5 g capacity and a resolution of 1 μg. The weighing chamber is a glass tube with a removable glass top. We replace the original top with one made of an aluminum alloy, as shown in Fig. 6. The alloy we use, designated AU4G in France, corresponds closely to ASTM 2017 and is the aluminum alloy commonly stocked by our workshop.

![Top View](image)

**Fig. 6.** Replacement cover for the balance, made by gluing an annular piece of aluminum alloy (AU4G) to the bottom of a thin disc of the same material. Dimensions are in millimeters.

9.2 Magnet

The cylindrical magnets used are made of neodymium-iron-boron (Vacuumschmelz GmbH, type
370 HR). The dimensions of each magnet are height = 2.5 mm; diameter = 5 mm. The axis of magnetization coincides with the geometric axis of the cylinder. The density of the NdFeB alloy is 7400 kg·m⁻³. Two such magnets are combined to produce a cylinder of height equal to diameter. In all equations given in the text, the origin of the coordinate system is the geometric center of the magnet.

When in doubt, the north pole of the magnet can easily be located by placing the magnet horizontally on a smooth, nonmagnetic surface. It then acts like a compass, spontaneously aligning itself along the Earth's horizontal field.

9.3 Pedestal

The magnet sits on a tubular pedestal of AU4G that is itself centered on the balance pan. Pedestal dimensions are shown in Fig. 7. Holes (not shown) drilled through the wall of the tube ensure that the total mass of the magnet and pedestal is well within the capacity of the balance.

An aluminum shim can be used to raise the magnet closer to the balance cover so that Z₀ is reduced to a minimum, thereby achieving maximum sensitivity. The shim is not used in routine work. The total mass of the magnets, pedestal and shim is about 4.2 g.

9.4 Bridge

Samples are centered on a bridge, also made of AU4G, that straddles the weighing chamber of the balance. Details are shown in Fig. 8. The important features of the design are that the span can be made level with respect to the balance pan and that the thickness of the span is the minimum consistent with adequate mechanical rigidity. If the span sags in the middle, samples of large diameter will be farther from the magnet than samples of small diameter. For critical measurements, a diamagnetic shim of small diameter can be placed at the center of the span thereby overcoming the problem of sag.

9.5 Gauge Blocks

We have found it convenient to place the bridge on nonmagnetic gauge blocks so that it may be raised or lowered in precise increments of 5 mm. We manufacture our blocks either of AU4G or of selected brass. The selection procedure consists of placing the brass stock in the vicinity where the eventual gauge block will be placed and verifying that there is no change in the balance reading with the magnet installed.

Fig. 7. Pedestal used to support the magnet. Not shown are 48 holes (eight columns of six), 5 mm in diameter, that are drilled into the tube in order to reduce its mass. The material used for the pedestal is AU4G; dimensions are in millimeters.

9.6 UMT5 Balance

A limited set of measurements was made using a model UMT5 balance (Mettler-Toledo AG), which has the same capacity and overall dimensions as the MT5 but has a resolution of 0.1 μg. The pan supplied with the UMT5 has a diameter of only 16 mm and, in order to use the pedestal shown in Fig. 7, the original pan was replaced by the pan from an MT5. Results obtained were satisfactory and indicate that the increased resolution allows the determination of the susceptibilities of samples of correspondingly smaller volumes.
variables are the radius $c$ and length $t$ of the cylindrical sample, $Z_0$, $\chi$, and $M_I$.

To find $Z_0$, we start with the measurement of $F_a$ made with the Alacrite standard. With $c$, $t$ and $\chi$, appropriate to this sample, we call upon a spreadsheet utility to compute the value of $Z_0$ through iterations of Eqs. (5a) and (6a), forcing the calculated value of $F_a$ to agree with that found by experiment. We check that $F_a$ with $M_I = 0$ is consistent with the computed result.

We then enter $c$ and $t$ appropriate to the sample under test and, calling once more on the spreadsheet utility, adjust $\chi$ until the measured value of $F_a$ is obtained. If necessary, we also adjust the value of $M_I$ until the calculated force $F_b$ agrees with experiment. A separate calculation for $\chi$ and $M_I$ is carried out for the sample in its normal orientation and turned upside down.

Typically, the sample under test has edges that are rounded with a radius of about 2 mm. To check whether the actual shape produces the same signal as a cylinder with sharp edges, we make supplementary calculations shown in Table 4.

The simplest calculation, which is usually sufficient, is to set limits on the susceptibility. That is, the susceptibility must be less than that computed assuming a perfect cylinder of radius $c$ and length $t$. The susceptibility must be greater than that computed for a perfect cylinder of radius $(c-2 \text{ mm})$ and length $t$. The results of these calculations are shown in Table 4. It is sufficient to calculate $I_s$ since $\chi$ is then found directly from Eq. (8).

We can also calculate the susceptibility more precisely. By superposition, we first determine $I_s$ due to the smallest annular volume that contains the lower rounded edge. The contribution is, in fact, small compared with the total $I_s$. We may be satisfied with this approximation; or we may note that all parts of the annulus are approximately equidistant from the magnet so that the signal from any arbitrarily chosen region of the annulus is roughly proportional to the volume of the region, as may be inferred from Eq. (5b). In particular, the material cut away from the annulus to make the rounded edge has a volume that is $(1-\pi/4)$ times the total annular volume. Thus we have underestimated $\chi$ by about 0.1%, an error that is well below the uncertainty of the final result. The correction for the upper rounded edge is even smaller, so we neglect it.

11. Appendix C. Typical Calculations for an OIML-Shaped Mass Standard

The mass standard is shown in Fig. 9. The lifting knob and the recessed base are complications that are dealt with in detail in this section. Two orientations of the sample are now considered.
Table 4. Computation of $\chi$ and $M_s$ for a 1 kg stainless steel standard of cylindrical shape. The height and diameter of the cylinder are both 54 mm and the edges are chamfered with a radius of 2 mm. Measurements were made with the cylinder coaxial to the magnet at a spacing of $Z_0 = 30.57$ mm.

<table>
<thead>
<tr>
<th>Experimental data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Normal</td>
</tr>
<tr>
<td>Reversed</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Calculation of $\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample dimensions</td>
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<tr>
<td>Height = diameter = 54 mm</td>
</tr>
<tr>
<td>Height = 54 mm; diameter = 50 mm</td>
</tr>
<tr>
<td>Height = diameter = 54 mm</td>
</tr>
<tr>
<td>Height = 2 mm; diameter = 54 mm</td>
</tr>
<tr>
<td>Height = 2 mm; diameter = 50 mm</td>
</tr>
<tr>
<td>Correction for lower edge</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Calculation of $M_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample orientation</td>
</tr>
<tr>
<td>Normal</td>
</tr>
<tr>
<td>Reversed</td>
</tr>
</tbody>
</table>

* $\pm 1 \mu N$ corresponds to a balance reading of $-101.9 \mu g$.

11.1 Sample Coaxial With the Magnet

The usual experimental orientation we use when determining magnetic susceptibility is coaxial with the magnet. We first measure $Z_0$ as described in Appendix B. It is then possible to put simple limits on the susceptibility: it must be less than that based on a sum and difference of "outer" cylinders that contain the entire volume of the standard; it must be greater than that based on a sum and difference of "inner" cylinders that are contained within the mass standard (see Fig. 9). The results are given in Table 5. Note that the net contribution of the knob is virtually negligible.

The same value of $Z_0$ was used in calculating each component of $I_s$. In general, any component $I_s' = I_s(Z_0')$ calculated using a different distance $Z_0'$ must be normalized by a factor $(Z_0/Z_0')^3$ before combining it with other components calculated using $Z_0$. The reason for this is easy to see: Consider an example where there are only two components in the summation, $I_s$ and $I_s'$, computed at distances $Z_0$ and $Z_0'$ respectively. From Eq. (5a) and the principle of superposition, the total measured force must be described by the relation

$$F_{total} = [\chi F_{max}(Z_0)] \cdot I_s + [\chi F_{max}(Z_0')] \cdot I_s' \cdot (9)$$

Fig. 9. Outer and inner cylinders used to approximate an OIML-shaped 1 kg mass standard by the method of superposition. Dimensions of the cylinders are given in Table 5.
Table 5. Computation of \( \chi \) and \( M_s \) for an OIML-shaped 1 kg stainless steel standard. Details of the shape are given in Fig. 9. Measurements were made with the cylinder coaxial to the magnet at a spacing of \( Z_0 = 24.70 \) mm.

<table>
<thead>
<tr>
<th>Calculation of ( \chi )</th>
<th>( c/\text{mm} )</th>
<th>( r/\text{mm} )</th>
<th>( l_s )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer cylinders</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>24.0</td>
<td>58.5</td>
<td>0.8196</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>16.5</td>
<td>1.0</td>
<td>0.1116</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>21.5</td>
<td>80.5</td>
<td>0.7690</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>21.5</td>
<td>58.5</td>
<td>-0.7681</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>0.7089</td>
<td>0.00364</td>
</tr>
<tr>
<td>Inner cylinders</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>24.0</td>
<td>58.5</td>
<td>0.8196</td>
<td></td>
</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>3</td>
<td>13.5</td>
<td>80.5</td>
<td>0.4979</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>13.5</td>
<td>58.5</td>
<td>-0.4975</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>0.7036</td>
<td>0.00367</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Calculation of ( M_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample dimensions</td>
</tr>
<tr>
<td>OIML</td>
</tr>
<tr>
<td>Cylinder, diam. = 48 mm; ht. = 59 mm</td>
</tr>
<tr>
<td>correction for cylindrical recess</td>
</tr>
<tr>
<td>Correction for bevel</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

*1 \( \mu \text{N} \) corresponds to a balance reading of \(-101.9 \mu \text{g}\).

By simple algebra,

\[
F_{\text{total}} = \left[ \chi^{\prime} F_{\text{max}} (Z_0) \right] \cdot \left[ I_s + \left( \frac{Z_0}{Z_0^*} \right)^4 I_s \right].
\]  

Equation (10) can easily be generalized to an arbitrary number of components.

Once the total \( I_s \) has been found, we can determine \( \chi \) from Eqs. (4) and (5a). Since \( F_0 \) is a relatively small number, we may approximate the sample as a cylinder resting on an annulus for this calculation. A more detailed calculation, shown in Table 5, verifies this assertion.

11.2 Axis of Sample Perpendicular to Axis of Magnet

It is possible to measure the susceptibility of a sample laid on its side, its body centered about the magnet axis. Our approach is, again, to calculate \( I_s \) and then solve for \( \chi \) using Eqs. (4) and (5a). In this case, however, \( I_s \) is determined from Eq. (5b) by numerical differentiation of the volume integral, after first converting to Cartesian coordinates. The computation can be done using packaged software such as Mathcad:

\[
Z_{0s} = 1
\]

\[
e = 10^{-8}
\]

\[
I_s = -\frac{8}{3 \pi} \int_0^{2\pi} \int_0^{Z_{0s}} \int_0^{\sqrt{r^2 - (Z_{0s} + z - Z_0)^2}} r^2 e^{-2z^2 + x^2 + y^2} \left( x^2 + y^2 + z^2 \right) dx dy dz.
\]

All integration limits are dimensionless, having first been normalized to \( Z_0 = 25.39 \) mm. The small quantity \( e \) is added to ensure that the upper limit of the second integral is real. Integration is over the dummy variables \( x, y \) and \( z \) representing the two horizontal axes and the vertical axis of a cartesian coordinate system where the...
Table 6. Calculations for horizontal orientations of the mass standard

<table>
<thead>
<tr>
<th>c/mm</th>
<th>t/mm</th>
<th>( l_1 )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>59</td>
<td>0.661</td>
<td>0.00368</td>
</tr>
<tr>
<td>24</td>
<td>21 + 59 + 21</td>
<td>0.684</td>
<td>0.00355</td>
</tr>
<tr>
<td>24</td>
<td>21 + 59</td>
<td>0.672</td>
<td>0.00362</td>
</tr>
</tbody>
</table>

The x-axis is parallel to the sample axis. Computing time can be shortened considerably by analytical integration over the x-variable, a result that is found in standard references on integral calculus. The results of the calculation are shown in Table 6. The measured value of the attractive force was \( F_s = 0.871 \) \( \mu \text{N} \). The first calculation excludes the knob in the calculation of \( l_1 \) and therefore overestimates \( \chi \). The second assumes there is a knob on each end of the body of the standard and that each knob is a simple extension of the body by 20 mm; this underestimates \( \chi \). The final calculation (the average of the previous two) assumes a single knob but models it again as a simple extension of the body. This again overestimates \( \chi \) but gives a tighter bound. Therefore 0.00368 > \( \chi \) > 0.00362.

If the sample were permanently magnetized perpendicular to its axis of symmetry, the measurements of \( F_s \) would depend on the azimuth of the sample and this can easily be checked. It is, of course, possible to estimate by numerical methods the expected value of \( F_s \) due simply to the Earth's field.

The measurements summarized in Tables 5 and 6 were made on different 1 kg standards fabricated from the same alloy.

**Acknowledgment**

The author thanks Mettler-Toledo AG for assistance in several aspects of this work.

**12. References**


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